Sketch of solution to Homework 2

Q1 If $x_n \to x$, then $\forall \epsilon > 0$, there is N such that for all $n > N$, $|x_n - x| < \epsilon$. Hence for any subsequence x_{n_k}, if $k > N$, $n_k > N$ which implies

$$|x_{n_k} - x| < \epsilon.$$

If every subsequence x_{n_k} has a subsequence that converges to x, if x_n does not converge to x, then there is $\epsilon_0 > 0$ and a subsequence x_{n_k} such that for all k,

$$|x_{n_k} - x| \geq \epsilon_0.$$

By bolzano weierstrass, there is a subsequence of x_{n_k} convergent to x. Hence we obtain contradictions.

Q3 If $\limsup x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k = \infty$, then for all $M > 0$, there is N such that for all $n > N$,

$$\sup_{k\geq n} x_k > M.$$

Then there is $k_n \geq n$ such that $x_{k_n} > M$.

Reversely, for Δ and $n \in \mathbb{N}$, there is $k_n \geq n$ such that $x_k > \Delta$. Hence, $\sup_{k\geq n} x_k > \Delta$. Since n is arbitrary, then we have the result.

Q5 For any n,

$$x_n + \inf_{k\geq n} y_k \leq x_n + y_n.$$

Then the first inequality follows from taking $\limsup$ both sides. The second inequality follows by considering

$$x_n + y_n \leq x_n + \sup_{k\geq n} y_k.$$

Remark: $\limsup$ coincides with $\lim$ if exists.

Q7 If x is a point of closure of E, then for all $\epsilon > 0$,

$$B(x,\epsilon) \cap E \neq \emptyset.$$

For $\epsilon = 1/n$, take $y_n \in E \cap B(x, 1/n)$. Hence, $y_n \to x$.

Suppose there is some sequence $y_n \in E$ such that $x = \lim_n y_n$. For any $\epsilon > 0$, there is N such that for all $n > N$, $y_n \in B(x,\epsilon)$.